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INEQUALITIES IN BICENTRIC QUADRILATERAL

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Chapter 1

Preliminary notions

Definition 1.1 A bicentric quadrilateral is a convex quadrilateral that has both an incircle and a circumcircle. The radius of these circles are called circumradius and irradius. We denote their lengths with R and r, with O the center of circumcircle and I the center of incircle. Bicentric quadrilateral respects all the properties of tangential and cyclic quadrilaterals.

Poncelet porism. If two circles one within the other are the incircle and circumcircle of a bicentric quadrilateral, then all the points of circumcircle are the vertices of a bicentric quadrilateral, having the same incenter and circumcircle.

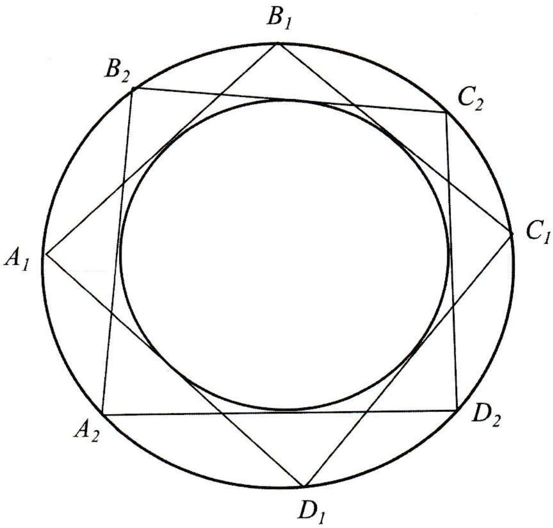

Figure 1.1

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If we denote with $\overline{d} = OI$ and if $\overline{d}^2 = R^2 + r^2 - r\sqrt{4R^2 + r^2}$ (Durrande-Fuss), then this is a sufficient condition to exist a quadrilateral with the incenter $\mathcal{C}(I, r)$ and the circumcenter $\mathcal{C}(O, R)$.
We denote $t_1 = AM$, $t_2 = MB$, $t_3 = CP$, $t_4 = DQ$.
We have $UM^2 + (MI - OU)^2 = \overline{d}^2$ or

$$\frac{(t_1 - t_2)^2}{t_1} + r^2 - 2rOU + R^2 - \frac{(t_1 + t_2)^2}{4} = R^2 + r^2 - r\sqrt{4R^2 + r^2}$$

or

$$r\sqrt{4R^2 + r^2} - t_1 t_2 = r\sqrt{4R^2 - (t_1 + t_2)^2}.$$

After squaring we obtain

$$2rt_1t_2\sqrt{4R^2 + r^2} = r^4 + r^2(t_1 + t_2)^2 + t_1^2 t_2^2.$$

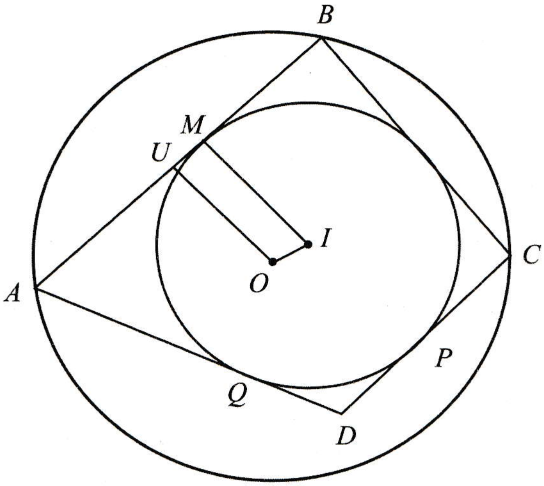

Figure 1.2

Similarly

$$2rt_2t_3\sqrt{4R^2 + r^2} = r^4 + r^2(t_2 + t_3)^2 + t_2^2 t_3^2.$$

After we divided the two equalities and perform some calculation, we obtain

$$r^4 + (t_2^2 - t_1 t_3)r^2 - t_1 t_2^2 t_3 = 0$$

or

$$(r^2 - t_1 t_3)(r^2 + t_2^2) = 0$$

or $\quad t_1 t_3 = r^2$. So $D \in \mathcal{C}(O, R)$.

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In the following we denote by $a = AB$, $b = BC$, $c = CD$, $d = DA$, $\overline{d} = OI$, s the semiperimeter, R the radius, r the inradius, F the area, $x_1 = bc + ad$, $x_2 = ab + cd$, $x_3 = ac + bd$, $BD = d_1$, $AC = d_2$. We have the following lemma:

Lemma 1.1 *In every bicentric quadrilateral the following equalities are true:*

$$F^2 = (s-a)(s-b)(s-c)(s-d) = abcd \tag{1.1}$$

$$x_1 + x_2 = s^2 \tag{1.2}$$

$$x_1 = \frac{4sRr}{d_1}, \quad x_2 = \frac{4sRr}{d_2} \tag{1.3}$$

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{s}{4Rr} \tag{1.4}$$

$$x_1 x_2 x_3 = 16R^2 r^2 s^2 \tag{1.5}$$

$$x_3 = 2r\left(\sqrt{4R^2 + r^2} + r\right) \tag{1.6}$$

$$(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2$$
$$= (a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2 \tag{1.7}$$

$$(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2$$
$$= 16r^4 s^2 \left[s^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)\right]\left[s^2 - \left(r + \sqrt{4R^2 + r^2}\right)\right]^2 \tag{1.8}$$

$$d_1 + d_2 = \frac{s}{2R}\left(\sqrt{4R^2 + r} + r\right) \tag{1.9}$$

$$R = \frac{1}{4}\sqrt{\frac{(ab + cd)(ad + bc)(ac + bd)}{abcd}} \tag{1.10}$$

$$r = \frac{\sqrt{abcd}}{a + c} = \frac{\sqrt{abcd}}{a + b} \tag{1.11}$$

$$\cos A = \frac{bc - ad}{bc + ad} \tag{1.12}$$

$$\sin\frac{A}{2} = \sqrt{\frac{ad}{ad + bc}} \tag{1.13}$$

$$\cos\frac{A}{2} = \sqrt{\frac{bc}{ad + bc}} \tag{1.14}$$

$$\tan\frac{A}{2} = \sqrt{\frac{ad}{bc}}\,. \tag{1.15}$$

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Proof.

(1) From Bretschneider's formula we have

$$F = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right)}$$

But since $ABCD$ is cyclic, we obtain $A + C = \pi$ or

$$F = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

which has called Brahmagupta's formula.
Since $ABCD$ is tangential we have $a + c = b + d$ or $s - a = c$, $s - b = d$, $s - c = a$, $s - d = b$. So $F = \sqrt{abcd}$.

(2) $x_1 + x_2 = bc + ad + ab + cd = (a+c)(b+d) = s^2$.

(3) We have $\sigma[ABD] + \sigma[BDC] = \sigma[ABCD] = \sigma[ABC] + \sigma[ADC]$ or $\begin{cases} ad\sin A + bc\sin C = 2F \\ ab\sin B + dc\sin D = 2F \end{cases}$, or $x_1 \sin A = 2sr$ and $x_2 \sin B = 2sr$, or $x_1\dfrac{d_1}{2R} = 2sr$ and $x_2\dfrac{d_2}{2R} = 2sr$, or $x_1 = \dfrac{4sRr}{d_1}$, $x_2 = \dfrac{4sRr}{d_2}$.

(4) It follows from 2) by adding the equality (1.3).

(5) It follows by multiplying the equalities from (1.3), taking into account from Ptolemy theorem that $d_1 d_2 = x_3$.

(6) We have

$$\begin{aligned} x_1x_2 &= (ab+cd)(ad+bc) = a^2bd + ab^2c + acd^2 + bc^2d \\ &= ac(b^2+d^2) + bd(a^2+c^2) = ac(s^2-2bd) + bd(s^2-2ac) \\ &= s^2x_3 - 4r^2s^2 = s^2(x_3 - 4r^2) \end{aligned}$$

But from (1.5) we obtain

$$\begin{aligned} & s^2(x_3 - 4r^2)x_3 = 16R^2r^2s^2 \quad \text{or} \\ & x_3^2 - 4r^2x_3 - 16R^2r^2 = 0 \quad \text{or} \\ & x_3 = 2r\left(\sqrt{4R^2+r^2} + r\right) \end{aligned}$$

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Another way to prove the equality is to use the equality

$$
\begin{aligned}
& s^2 = (s-a)(s-b)(s-c)(s-d) = s^2r^2 \qquad \text{or} \\
& -s^3 + \sigma_2 s - \sigma_3 = 0, \qquad \text{or} \\
& \sigma_3 = s(\sigma_2 - s^2), \qquad \text{or} \\
& s^2(\sigma_2 - s^2)^2 - 4s^2r^2\sigma_2 + 4r^2s^4 = 16R^2r^2s^2, \qquad \text{or} \\
& \sigma_2^2 - (2s^2 + 4r^2)\sigma_2 + s^4 + 2s^2r^2 - 16r^2R^2 = 0, \qquad \text{or} \\
& \sigma_2 = s^2 + 2r^2 + 2r\sqrt{4R^2 + r^2}.
\end{aligned}
$$

Since $\sigma_2 = x_1 + x_2 + x_3 = s^2 + x_3$, it follows the statement.

(7) We have

$$
\begin{aligned}
&(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2 \\
&= [(a-b)(c-d)]^2 [(a-c)(b-d)]^2 [(a-d)(b-c)]^2 \\
&= (x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2.
\end{aligned}
$$

(8) From (1.7) we have

$$
\begin{aligned}
\Pi(a-b)^2 &= (x_1 - x_2)^2(x_3 - x_1)^2(x_3 - x_2)^2 \\
&= \left[(x_1 + x_2)^2 - 4x_1x_2\right] \left[x_3^2 - x_3(x_1 + x_2) + x_1x_2\right]^2 \qquad (*)
\end{aligned}
$$

But from (1.5) and (1.6) we have

$$
x_1x_2 = \frac{16R^2r^2s^2}{2r\left(\sqrt{4R^2 + r^2} + r\right)} = 2r\left(\sqrt{4R^2 + r^2} - r\right)s^2
$$

and $x_1 + x_2 = s^2$.
Replacing in $(*)$ we obtain

$$
\begin{aligned}
\Pi(a-b)^2 = &\left[s^4 - 8r\left(\sqrt{4R^2 + r^2} - r\right)s^2\right]\left[4r^2\left(r + \sqrt{4R^2 + r^2}\right)^2\right. \\
&\left. - 2r\left(r + \sqrt{4R^2 + r^2}\right)s^2 + 2r\left(\sqrt{4R^2 + r^2} - r\right)s^2\right]^2 \\
&= 16r^4s^2\left[s^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)\right]\left[s^2 - \left(r + \sqrt{4R^2 + r^2}\right)^2\right]^2.
\end{aligned}
$$

(9) It follows from (1.4) and (1.6) since $d_1d_2 = x_3$.

(10) It follows from (1.5) and $F = sr$ or $s^2r^2 = abcd$.

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(11) We have $r = \dfrac{F}{s} = \dfrac{\sqrt{abcd}}{a+c}$.

(12) From cosine theorem in triangles ABD and BCD we obtain

$$\begin{cases} 2ad\cos A & = a^2 + d^2 - d_1^2 \\ -2bc\cos A & = b^2 + c^2 - d_1^2 \end{cases}.$$

Decreasing these equalities, we obtain

$$\cos A = \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)} = \frac{bc - ad}{bc + ad}.$$

(1.13), (1.14) and (1.15) it follows from (1.12) and formulas

$$\sin\frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}}, \quad \cos\frac{A}{2} = \sqrt{\frac{bc}{ad + bc}}, \quad \tan\frac{A}{2} = \sqrt{\frac{ad}{bc}}.$$

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Chapter 2

Blundon-Eddy inequality

2.1 A geometrical proof of Blundon-Eddy inequality in bicentric quadrilateral

The purpose of this chapter is to give a geometrical proof of Blundon-Eddy inequality using basic knowledge of mathematical analysis.

Let $ABCD$ a bicentric quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$, O the center of circumcenter with radius R, I the center of incenter with radius r, S the semiperimeter, $\overline{d} = OI$, $x_1 = bc + ad$, $x_2 = ab + cd$, $x_3 = ac + bd$. We remember the following lemma.

Lemma 2.1 *In every bicentric quadrilateral the following equalities are true:*

i) $x_1 + x_2 = S^2$

ii) $x_1 x_2 x_3 = S^2 + 2r^2 + 2r\sqrt{4R^2 + r^2}$

iii) $x_3 = 2r\left(\sqrt{4R^2 + r^2} + r\right)$

iv) $\tan^2 \dfrac{A}{2} = \dfrac{bc}{ad}$

Proof. *i)*, *ii)* and *iii)* are proved in [2], *iv)* see [6].

Also, the Fuss theorem is well known.

Theorem 2.1 *In every bicentric quadrilateral the following equalities are true:*

i) $\overline{d}^2 = R^2 + r^2 - r\sqrt{4R^2 + r^2}$

ii) $\dfrac{1}{(R - \overline{d})^2} + \dfrac{1}{(R + \overline{d})^2} = \dfrac{1}{r^2}$

Proof. See [3], [4] and [5].

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In the following we give a new proof of Blundon-Eddy inequality.

Theorem 2.2 *In every bicentric quadrilateral $ABCD$ the inequality $S_1 \le S \le S_2$ is true, where $S_1 = \sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}$ represents the semiperimeter of bicentric quadrilateral $A_1B_1C_1D_1$ with the sides:*

$$\{a_1, b_1, c_1, d_1\} = \Big\{2\sqrt{R^2-(r+\overline{d})^2},\ \sqrt{R^2-(r-\overline{d})^2}+\sqrt{R^2-(r+\overline{d})^2},$$
$$2\sqrt{R^2-(r-\overline{d})^2},\ \sqrt{R^2-(r-\overline{d})^2}+\sqrt{R^2-(r+\overline{d})^2}\Big\}$$

and $S_2 = \sqrt{4R^2+r^2}+r$ the semiperimeter of bicentric quadrilateral $A_2B_2C_2D_2$ with the sides

$$\{a_2, b_2, c_2, d_2\} = \left\{\frac{2R}{R-\overline{d}}\sqrt{(R-\overline{d})^2-r^2},\ \frac{2R}{R+\overline{d}}\sqrt{(R+\overline{d})^2-r^2},\right.$$
$$\left.\frac{2R}{R+d}\sqrt{(R+\overline{d})^2-r^2},\ \frac{2R}{R-\overline{d}}\sqrt{(R-\overline{d})^2-r^2}\right\}$$

Proof. Let $\mathcal{C}(O,R)$, $\mathcal{C}(I,r)$ with R, r fixed and S variable which satisfies the Fuss theorem.
We consider the vertical diameter A_2C_2 such that A_2, I, O, C_2 be collinear.

Let $A \in \mathcal{C}(O,R)$ a variable point situated on the left semiplane in relation to A_2C_2 and $B, C, D \in \mathcal{C}(O,R)$ such that AB, BC, CD are tangent to $\mathcal{C}(I,r)$.

According to Poncelet theorem, A_2 is tangent to $\mathcal{C}(I,r)$.

We denote $\alpha = \mu(\widehat{AOA_2})$, $\alpha \in [0,\pi]$

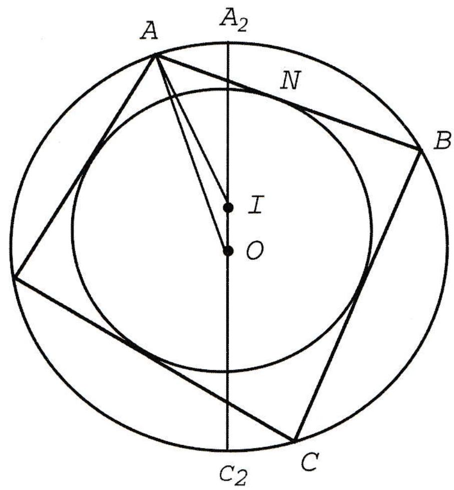

Figure 2.1